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两个 loop 代数及其应用

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摘 要: 基于两个 loop 代数,利用(2+1)-维零曲率方程分别得到了(2+1)-维超 AKNS 族和超 Tu 族.

关键词: loop 代数; 屠格式; (2+1)-维零曲率方程

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Two Super-loop Algebras and Its Application

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Abstract: This paper respectively obtained the (2+1)-dimensional super-AKNS hierarchy and (2+1)-dimensional super-Tu hierarchy with (2+1)-dimensional zero curvature equation based on two super-loop algebras.

Keywords: Super-loop algebra; Tu scheme; (2+1)-dimensional zero curvature equation

寻找尽可能多的可积孤立子方程族是孤立子理论研究中的一项重要而有趣的课题,目前流行的方法是屠格式,人们利用屠格式获得了许多有重要物理意义的可积方程族,如 AKNS,KN,BPT 族等[1-3];随着可积系统研究的不断深入,(2+1)-维可积和超可积系统引起了研究兴趣[4-9].在文献[10]中周子翔教授引入了如下的 Lax 对

$$\begin{cases} \varphi_y = A\varphi_x + U\varphi, \\ \varphi_t = B\varphi_x + V\varphi, \lambda_t = 0. \end{cases} \tag{1}$$

其相容性条件为

$$\begin{cases} U_t - V_y + [U, V] + AV_x - BU_x = 0, \\ [A, V] = [B, U], \end{cases} \tag{2}$$

其中 $A = diag(a_1, a_2, \ldots, a_n)$, $B = diag(b_1, b_2, \ldots, b_n)$. 在本文中,我们利用屠格式寻找(2+1)-维的超可积系统.

1 (2+1)维超 AKNS 方程族

在文献[6]中,马文秀教授给出了如下的 loop 代数 B(0,1)

$$\begin{cases} E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, [E_1, E_2] = 2E_2, [E_1, E_3] = -2E_3, [E_2, E_3] = E_1, \\ [E_1, E_4] = E_4, [E_1, E_5] = -E_5, [E_2, E_4] = 0, [E_2, E_5] = E_4, [E_3, E_4] = E_5, \\ [E_3, E_5] = 0, [E_4, E_5]_+ = E_1, [E_4, E_4]_+ = -2E_2, [E_5, E_5]_+ = 2E_3. \end{cases} \tag{3}$$

其中 E_1, E_2, E_3 是偶元,E_4,E_5 是奇元,符号[,]和[,]分别表示交换子和反交换子.相应的 loop 代数 $\tilde{B}(0,1)$ 为

$$\begin{cases} E_1(n) = \lambda^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2(n) = \lambda^n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_3(n) = \lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ E_4(n) = \lambda^n \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, E_5(n) = \lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, [E_1(m), E_2(n)] = 2E_2(m+n), \\ [E_1(m), E_3(n)] = -2E_3(m+n),\ [E_2(m), E_3(n)] = E_1(m+n), [E_1(m), E_4(n)] = E_4(m+n), \\ [E_1(m), E_5(n)] = -E_5(m+n), [E_2(m), E_4(n)] = 0, [E_2(m), E_5(n)] = E_4(m+n), \\ [E_3(m), E_4(n)] = E_5(m+n), [E_3(m), E_5(n)] = 0, [E_4(m), E_5(n)]_+ = E_1(m+n), \\ [E_4(m), E_4(n)]_+ = -2E_2(m+n), [E_5(m), E_5(n)]_+ = 2E_3(m+n). \end{cases} \tag{4}$$

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考虑如下的(2+1) 维超等谱问题

$$\begin{cases}\varphi_y = \varphi_x + U\varphi,, \lambda_t = 0\\ U = E_1(1) + pE_2(0) + qE_3(0) + rE_4(0) + sE_5(0).\end{cases} \tag{5}$$

取 $$V = \sum_{m=0}^{\infty}(a_m E_1(-m) + b_m E_2(-m) + c_m E_3(-m) + d_m E_4(-m) + f_m E_5(-m)), \tag{6}$$

解静态零曲率方程 $V_y - \alpha V_x = [U, V]$, 得

$$\begin{cases} a_{my} - \alpha a_{mx} = pc_m - qb_m + rf_m + sd_m,\\ b_{my} - \alpha b_{mx} = 2b_{m+1} - 2pa_m - 2rd_m,\\ c_{my} - \alpha c_{mx} = -2c_{m+1} + 2qa_m + 2sf_m,\\ d_{my} - \alpha d_{mx} = d_{m+1} - ra_m + pf_m - sb_m,\\ f_{my} - \alpha f_{mx} = -f_{m+1} + qd_m + sa_m - rc_m.\end{cases} \tag{7}$$

记 $$\begin{cases} V_+^{(n)} = \sum_{m=0}^{n}(a_m E_1(n-m) + b_m E_2(n-m) + c_m E_3(n-m) + d_m E_4(n-m) + f_m E_5(n-m)),\\ V_-^{(n)} = \lambda^n V - V_+^{(n)}.\end{cases} \tag{8}$$

A 直接计算得 $$-V_{+y}^{(n)} + \alpha V_{+x}^{(n)} + [U, V_+^{(n)}] = 2b_{n+1}E_2(0) - 2c_{n+1}E_3(0) + d_{n+1}E_4(0) - f_{n+1}E_5(0). \tag{9}$$

取 $V^{(n)} = V_+^{(n)}$,

则由零曲率方程 $$U_t - V_y^{(n)} + [U, V^{(n)}] + \alpha V_x^{(n)} - \beta U_x = 0 \tag{10}$$

可得下面的 (2+1)-维超可积系统

$$u_t = \begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix}_t = \beta\begin{pmatrix}\partial_x & 0 & 0 & 0\\ 0 & \partial_x & 0 & 0\\ 0 & 0 & \partial_x & 0\\ 0 & 0 & 0 & \partial_x\end{pmatrix}\begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix} + \begin{pmatrix} 0 & 2 & 0 & 0\\ -2 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix} -c_{n+1}\\ -b_{n+1}\\ 2f_{n+1}\\ -2d_{n+1}\end{pmatrix} \tag{11}$$

$$= \beta J_1\begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix} + J_2 P_{n+1}.$$

由递推关系 (7) 可得递推算子 L 为

$$P_{n+1} = LP_n, L = \begin{pmatrix} L_{11} & q(\alpha\partial_x^{-1} - \partial_y^{-1})q & L_{13} & L_{14}\\ p(\partial_y^{-1} - \alpha\partial_x^{-1})p & L_{22} & -\frac{1}{2}p(\partial_y^{-1} - \alpha\partial_x^{-1})r & L_{24}\\ L_{31} & 2s(\partial_y^{-1} - \alpha\partial_x^{-1})q & L_{33} & L_{34}\\ 2r(\partial_y^{-1} - \alpha\partial_x^{-1})p & L_{42} & -p + r(\partial_y^{-1} - \alpha\partial_x^{-1})r & L_{44}\end{pmatrix}, \tag{12}$$

其中 $L_{11} = \frac{1}{2}(\alpha\partial_x - \partial_y) + q(\partial_y^{-1} - \alpha\partial_x^{-1})p, L_{31} = 2r - 2s(\partial_y^{-1} - \alpha\partial_x^{-1})p,$

$L_{22} = \frac{1}{2}(\partial_y - \alpha\partial_x) - p(\partial_y^{-1} - \alpha\partial_x^{-1})q, L_{42} = -2s + 2r(\partial_y^{-1} - \alpha\partial_x^{-1})q,$

$L_{13} = -\frac{1}{2}s - \frac{1}{2}q(\partial_y^{-1} - \alpha\partial_x^{-1})r, L_{33} = \alpha\partial_x - \partial_y + s(\partial_y^{-1} - \alpha\partial_x^{-1})r,$

$L_{14} = \frac{1}{2}q(\partial_y^{-1} - \alpha\partial_x^{-1})s, L_{24} = \frac{r}{2} + \frac{1}{2}p(\partial_y^{-1} - \alpha\partial_x^{-1})s,$

$L_{34} = -q - s(\partial_y^{-1} - \alpha\partial_x^{-1})s, L_{44} = \alpha\partial_x - \partial_y - r(\partial_y^{-1} - \alpha\partial_x^{-1})s.$

因此, (11) 可化为如下形式

$$u_t = \beta J_1\begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix} + J_2 L^n\begin{pmatrix} -c_1\\ -b_1\\ 2f_1\\ -2d_1\end{pmatrix}. \tag{13}$$

在 (13) 中取 $\partial_x^{-1} = \partial_x = 0$, 则 (13) 可约化为超 AKNS 族.

2 (2+1)维超 Tu 族

作 $E_1,E_2,\ldots,E_5$ 的线性组合, 则可得如下的 loop 代数 $B_1(0,1)$

$$\begin{cases} e_1=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, e_2=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, e_3=\frac{1}{2}\begin{pmatrix}0&1&0\\-1&0&0\%\\0&0&0\end{pmatrix}, \\ e_4=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix}, e_5=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}, [e_1,e_2]=e_3, [e_1,e_3]=e_2, \\ [e_3,e_2]=e_1, [e_1,e_4]=[e_2,e_5]=[e_3,e_5]=\frac{e_4}{2}, [e_5,e_1]=[e_2,e_4]=[e_4,e_3]=\frac{e_5}{2}, \\ [e_4,e_5]_+=[e_5,e_4]_+=\frac{e_1}{2}, [e_4,e_4]_+=-\frac{e_2+e_3}{2}, [e_5,e_5]_+=\frac{e_2-e_3}{2}. \end{cases} \tag{14}$$

相应的超圈代数 $\tilde{B}_1(0,1)$ 为

$$\begin{cases} e_1(n)=\frac{1}{2}\lambda^n\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, e_2(n)=\frac{1}{2}\lambda^n\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, e_3(n)=\frac{1}{2}\lambda^n\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}, \\ e_4(n)=\frac{1}{2}\lambda^n\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix}, e_5(n)=\frac{1}{2}\lambda^n\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}, [e_1(m),e_2(n)]=e_3(m+n), \\ [e_1(m),e_3(n)]=e_2(m+n), [e_3(m),e_2(n)]=e_1(m+n), [e_1(m),e_4(n)]=\frac{e_4(m+n)}{2}, \\ [e_2(m),e_5(n)]=\frac{e_4(m+n)}{2}, [e_3(m),e_5(n)]=\frac{e_4(m+n)}{2}, [e_5(m),e_1(n)]=\frac{e_5(m+n)}{2}, \\ [e_2(m),e_4(n)]=\frac{e_5(m+n)}{2}, [e_4(m),e_3(n)]=\frac{e_5(m+n)}{2}, [e_5(m),e_5(n)]_+=\frac{e_2(m+n)-e_3(m+n)}{2}, \\ [e_4(m),e_4(n)]_+=-\frac{e_2(m+n)+e_3(m+n)}{2}, [e_4(m),e_5(n)]_+=[e_5(m),e_4(n)]_+=\frac{e_1}{2}(m+n). \end{cases} \tag{15}$$

考虑如下的 (2+1) 维超等谱问题

$$\begin{cases} \varphi_y=\varphi_x+U\varphi, \lambda_t=0, \\ U=-2e_1(1)+u_1e_1(0)+2u_2e_2(0)+u_3e_4(0)+u_4e_5(0). \end{cases} \tag{16}$$

取
$$V=\sum_{m=0}^{\infty}(a_me_1(-m)+b_me_2(-m)+c_me_3(-m)+d_me_4(-m)+f_me_5(-m)), \tag{17}$$

解静态零曲率方程 V_y-αV_x=[U,V] 可得

$$\begin{cases} a_{my}-\alpha a_{mx}=-2u_2c_m+\frac{1}{2}u_3f_m+\frac{1}{2}u_4d_m, \\ b_{my}-\alpha b_{mx}=u_1c_m-2c_{m+1}+\frac{1}{2}u_4f_m-\frac{1}{2}u_3d_m, \\ c_{my}-\alpha c_{mx}=u_1b_m-2b_{m+1}-2u_2a_m-\frac{1}{2}u_3d_m-\frac{1}{2}u_4f_m, \\ d_{my}-\alpha d_{mx}=-d_{m+1}+\frac{1}{2}u_1d_m+u_2f_m-\frac{1}{2}u_3a_m-\frac{1}{2}u_4b_m-\frac{1}{2}u_4c_m, \\ f_{my}-\alpha f_{mx}=f_{m+1}-\frac{1}{2}u_1f_m+u_2d_m+\frac{1}{2}u_4a_m-\frac{1}{2}u_3b_m+\frac{1}{2}u_3c_m. \end{cases} \tag{18}$$

记
$$\begin{cases} V_+^{(n)}=\sum_{m=0}^{n}(a_me_1(n-m)+b_me_2(n-m)+c_me_3(n-m)+d_me_4(n-m)+f_me_5(n-m)), \\ V_-^{(n)}=\lambda^nV-V_+^{(n)}. \end{cases} \tag{19}$$

直接计算可得
$$-V_{+y}^{(n)}+\alpha V_{+x}^{(n)}+[U,V_+^{(n)}]=2c_{n+1}e_2(0)+2b_{n+1}e_3(0)+d_{n+1}e_4(0)-f_{n+1}e_5(0). \tag{20}$$

取 $V^{(n)}=V_+^{(n)}+\Delta_n, \Delta_n=\frac{b_{n+1}}{u_2}e_1(0)$, 则由零曲率方程

$$U_t - V_y^{(n)} + [U, V^{(n)}] + \alpha V_x^{(n)} - \beta U_x = 0 \tag{21}$$

可得如下的 (2+1)-维超可积系统

$$u_t - \beta u_x = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t - \beta \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_x = \begin{pmatrix} \left(\frac{b_{n+1}}{u_2}\right)_y - \alpha \left(\frac{b_{n+1}}{u_2}\right)_x \\ \frac{a_{n+1,y}}{2u_2} - \alpha \frac{a_{n+1,x}}{2u_2} - \frac{u_3}{4u_2} f_{n+1} - \frac{u_4}{4u_2} d_{n+1} \\ \frac{u_3}{2u_2} b_{n+1} - d_{n+1} \\ f_{n+1} - \frac{u_4}{2u_2} b_{n+1} \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} 0 & (\partial_y - \alpha\partial_x)\frac{1}{2u_2} & 0 & 0 \\ \frac{1}{2u_2}(\partial_y - \alpha\partial_x) & 0 & \frac{u_3}{4u_2} & -\frac{u_4}{4u_2} \\ 0 & \frac{u_3}{4u_2} & 0 & -1 \\ 0 & -\frac{u_4}{4u_2} & -1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ 2b_{n+1} \\ -f_{n+1} \\ d_{n+1} \end{pmatrix} = JK_{n+1}.$$

由递推关系 (18) 可得递推算子 L 为

$$K_{n+1} = LK_n, L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & \frac{u_1}{2} & \frac{u_4}{2} + (\partial_y - \alpha\partial_x)\frac{u_3}{4u_2} & -\frac{u_3}{2} + (\alpha\partial_x - \partial_y)\frac{u_4}{4u_2} \\ L_{31} & -\frac{u_3}{4} & \frac{u_1}{2} + \partial_y - \alpha\partial_x - \frac{u_3^2}{8u_2} & u_2 + \frac{u_3 u_4}{8u_2} \\ L_{41} & -\frac{u_4}{4} & -u_2 + \frac{u_3 u_4}{8u_2} & \frac{u_1}{2} + \alpha\partial_x - \partial_y - \frac{u_4^2}{8u_2} \end{pmatrix}, \tag{23}$$

其中

$$L_{11} = \frac{1}{8}(\partial_y^{-1} - \alpha\partial_x^{-1})\frac{u_3^2 + u_4^2}{u_2}(\partial_y - \alpha\partial_x) - \frac{1}{2}(\partial_y^{-1} - \alpha\partial_x^{-1})u_3 u_4 + \frac{1}{2}(\partial_y^{-1} - \alpha\partial_x^{-1})u_1(\partial_y - \alpha\partial_x),$$

$$L_{12} = \frac{1}{8}(\partial_y^{-1} - \alpha\partial_x^{-1})(u_3^2 - u_4^2) + \frac{1}{2}(\partial_y^{-1} - \alpha\partial_x^{-1})u_2(\partial_y - \alpha\partial_x),$$

$$L_{13} = \frac{1}{16}(\partial_y^{-1} - \alpha\partial_x^{-1})\frac{u_3}{u_2}(u_3^2 + u_4^2) - \frac{1}{2}(\partial_y^{-1} - \alpha\partial_x^{-1})u_3(\partial_y - \alpha\partial_x),$$

$$L_{14} = -\frac{1}{16}(\partial_y^{-1} - \alpha\partial_x^{-1})\frac{u_4}{u_2}(u_3^2 + u_4^2) - \frac{1}{2}(\partial_y^{-1} - \alpha\partial_x^{-1})u_4(\partial_y - \alpha\partial_x),$$

$$L_{21} = -2u_2 + (\partial_y - \alpha\partial_x)\frac{1}{2u_2}(\partial_y - \alpha\partial_x),$$

$$L_{31} = \frac{u_4}{2} + \frac{u_3}{4u_2}(\alpha\partial_x - \partial_y), L_{41} = -\frac{u_3}{2} + \frac{u_4}{4u_2}(\partial_y - \alpha\partial_x).$$

故 (22) 可化为如下形式

$$u_t - \beta u_x = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t - \beta \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_x = JL^n \begin{pmatrix} a_1 \\ 2b_1 \\ -f_1 \\ d_1 \end{pmatrix}. \tag{24}$$

在(24)中取 $\partial_x^{-1} = \partial_x = 0$, 则(24)可约化为超 Tu 族.

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